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ENDS OF GROUPS WITH THE INTEGERS AS QUOTIENT

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1. Introduction

This paper continues the study of semi-stability at ∞ for finitely presented groups begun in [10], and we continue to examine the question: If G is a finitely presented group, is $H^2(G,\mathbb{Z}G)$ free abelian? (See [4].)

Given X, a locally compact, separable metric space and $r,s:[0,\infty)\to X$ proper maps, then r and s converge to the same end of X if for each compact set $C\subset X$, there is an integer N such that $r([N,\infty))$ and $s([N,\infty))$ lie in the same unbounded path component of $X-C$. An end of X is an equivalence class of maps $[0,\infty)\to X$, where r and s are equivalent if they converge to the same end. X is semi-stable at ∞ if maps converging to the same end are properly homotopic. (See [10].) The (cardinal) number of ends, and the semi-stability at ∞ notions are defined for finitely presented groups. If X is a finite CW-complex with $\pi_1(X)=G$, then the number of ends of G is the number of ends of $\tilde{X}$, the universal cover of X, G is semi-stable at ∞ if $\tilde{X}$ is. These concepts are well-defined by the following theorem.

Theorem. *If K and L are finite CW-complexes and $\pi_1(K)=\pi_1(L)$, then $\tilde{K}$ and $\tilde{L}$ have the same number of ends, and $\tilde{K}$ is semi-stable at ∞ if and only if $\tilde{L}$ is.*

Proof. See the proof of Theorem 3 of [7].

By a theorem in [4], if a finitely presented group, G, is semi-stable at ∞, then $H^2(G:\mathbb{Z}G)$ is free abelian. In fact, if X is a finite CW-complex with $\pi_1(X)=G$, then $H^2(G:\mathbb{Z}G)$ will be free abelian iff the following condition is met: Given C a compact subset of $\tilde{X}$, there is a compact set $D\subset\tilde{X}$ such that for any compact set $E\subset\tilde{X}$ and loop α in $\tilde{X}-D$, α is homologous, in $\tilde{X}-C$, to a loop in $\tilde{X}-E$.

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The following questions are well known:

Question 1. Do all finitely presented groups, G, have free abelian second cohomology (with $\mathbb{Z}G$-coefficients)?

Question 2. Are all finitely presented groups semi-stable at ∞?

Question 2 is related to important questions in the study of 3-manifolds, such as: Are Whitehead's contractible 3-manifold and open 3-manifolds of similar construction [9, 14] universal covers of closed 3-manifolds? (This question is examined in [10].) The recent examples of Davis [3] in dimensions ≥ 4 do not settle Question 2.

Now we state our results. Let $N(A)$ denote the normal closure of the subset A of the group G.

Main Theorem. *Let G be a finitely presented group with generators $\langle x, h_1, \dots, h_n, k_1, \dots, k_m \rangle$. Let H and K be the subgroups of G, generated by $\langle h_1, \dots, h_n \rangle$ and $\langle k_1, \dots, k_m \rangle$ respectively. Assume:*

(i) *H and K are finitely presented.*

(ii) *$G/N(H \cup K) \cong \mathbb{Z}$.*

(iii) *$xHx^{-1} \subset H$ and $x^{-1}Kx \subset K$.*

Then G is either 1 or 2-ended and G is semi-stable at ∞.

Theorem 3.1. *If H is a 1-ended finitely presented group and $\phi: H \to H$ is a monomorphism, then the HNN-extension $[H : H, \phi]$ is simply connected at ∞.*

For the definition of HNN-extension see [8, p. 179]. A finitely presented group, G, is simply connected at ∞ if some (equivalently any) finite complex, X, with $\pi_1(X) = G$ has the property: Given a compact set, C, in $\tilde{X}$, there is a compact set $D \subset \tilde{X}$, such that any loop in $\tilde{X} - D$ is homotopically trivial in $\tilde{X} - C$. Theorems in [5] and [6] show that if the ϕ of Theorem 3.1 is an isomorphism, then G is simply connected at ∞.

Theorem 3.2. *If H_1 and H_2 are 1-ended finitely presented subgroups of the finitely presented group G such that H_1 and H_2 are semi-stable at ∞, $H_1 \cup H_2$ generates G and $H_1 \cap H_2$ contains a finitely generated infinite group, then G is semi-stable at ∞ and 1-ended.*

2. A question about knot groups

All knot groups (with the exception of the trivial knot) are 1-ended [11], with abelianization $\mathbb{Z}$. A Weirtinger presentation for a knot group, G, leads to a presentation of G with generators $x, g_1, g_2, \dots, g_n$: where $G/N\{g_1, \dots, g_n\} = \mathbb{Z}$. By Scott's theorem [13], finitely generated subgroups of compact 3-manifold groups are finitely presented. Hence, if one were to search for a knot group satisfying the hypothesis of our main theorem, condition (i) could be relaxed to: H and K are finitely generated. If the commutator subgroup, C, of a knot group, G is finitely generated, then the hypothesis of our theorem is satisfied with $H = K = C$. We have no example of a knot group, G, satisfying the hypothesis of our theorem or of one failing to satisfy the hypothesis of our theorem, if the commutator subgroup of G is not finitely generated. R.H. Crowell and E.M. Brown [2] prove an interesting theorem which implies the following: If G is a knot group with non-finitely generated commutator subgroup, C, then the hypothesis of our theorem cannot be satisfied with $N(H) = C$ or $N(K) = C$.

Such an example would be a so-called *lobster pot knot*. These knots cannot exist by [2]. Finally, a large list of knots with non-finitely generated commutator subgroup can be identified by a process described on p. 326 of [12].

3. A preliminary theorem

Theorem 3.1 is proved by using elementary ideas that also appear in the proof of the main theorem. Hence, it provides an introduction to the more sophisticated ideas and techniques of our main theorem. Now we provide a setting to prove our theorem in. Given a presentation $p = \langle g_1, \dots, g_n : r_n, \dots, r_m \rangle$ of a group G, one builds the standard 2-complex, X_p, with $\pi_1(X_p) = G$ as follows: There is a single vertex, $*$. For each generator, g_i, attach a loop at $*$. Now attach 2-cells to these loops according to the relations r_i. Let $\tilde{X}_p$ be the universal cover of X_p. $\tilde{X}_p$ can be constructed as follows: There is one vertex for each element of G, and an edge between vertices v and w if $vw^{-1} \in \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$. This is the 1-skeleton of $\tilde{X}_p$. 2-cells are attached according to the relations r_i (see [6]). The edges of $\tilde{X}_p$ correspond to the group elements $\langle g_1^{\pm 1}, \dots, g_n^{\pm 1} \rangle$. An edge path $\langle e_1, \dots, e_k \rangle$ of $\tilde{X}_p$ corresponds to $\langle e_1', \dots, e_k' \rangle$ where $e_i' \in \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ but to obtain a direct correspondence between edge paths and the corresponding list of generators, it is necessary to specify the initial point of $\langle e_1, \dots, e_k \rangle$, when referring to $\langle e_1', \dots, e_k' \rangle$, since at any vertex v, $\langle e_1', \dots, e_k' \rangle$ determines an edge path (that differs from $\langle e_1', \dots, e_k' \rangle$ at the vertex w by a covering transformation).

Theorem 3.1. *If H is a* 1*-ended finitely presented group and $\phi : H \to H$ is a monomorphism, then the HNN-extension $[H : H, \phi]$ is simply connected at ∞. (See* [8, *p.* 179] *for the definition of $[H : H, \phi]$.)*

Proof. Assume $\langle h_1, \dots, h_n : r_1, \dots, r_m \rangle$ is a presentation of H. Then the desired HNN-extension, G, has a presentation $\langle x, h_1, \dots, h_n : r_1, \dots, r_m, xh_ix^{-1} = \phi(h_i)$ for all $i \in \{1, \dots, n\}\rangle$ (here we assume $\phi(h_i)$ is a word in the letters $h_1^{\pm 1}, \dots, h_n^{\pm 1}$). Let X and

Y be the standard 2-complexes obtained from the above presentations of G and H respectively. By Britton's lemma (see [1, p. 20]) the map $\xi : H \to G$ defined by $\xi(h_i) = h_i$ for $i \in \{1, \ldots, n\}$ is a monomorphism. Y includes naturally into X, and since ξ is a monomorphism, this inclusion induces an injection of $\pi_1(Y)$ into $\pi_1(X)$. Hence for each element of G/H, there is a copy of $\tilde{Y}$, the universal cover of Y, in $\tilde{X}$, the universal cover of X. These copies of $\tilde{Y}$ are mutually disjoint. $N(H)$ is the normal closure of H in G, and $G/N(H) = \mathbb{Z}$. Let $*(0)$ be a vertex of $\tilde{X}$ and $*(n)$ be the end point of the lift of x^n to $*(0)$.

If v is a vetex of $\tilde{X}$, and $\pi : \tilde{X} \to \tilde{X}/N(H)$ is projection, then $\pi(v) = \pi(*(n))$ for some integer n, and we say v is in level n of $\tilde{X}$. Furthermore, if W is the copy of $\tilde{Y}$ containing v, then we say W is in level n of $\tilde{X}$. As base ray in $\tilde{X}$, we choose the proper edge path to ∞, $r : ([0, \infty), \{0\}) \to (\tilde{X}, *(0))$, such that r restricted to $[n, n+1]$ is the lift of the loop x to $*(n)$. Let e be an edge of $\tilde{X}$, with initial point $*(0)$, such that e corresponds to an element of $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$. Since $xex^{-1} = \phi(e)$, we say e can be slid along x to an edge path with edges in $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$. See Fig. 1

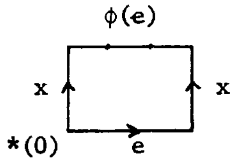

Fig. 1.

In $\tilde{X}$, a 2-cell is attached to the loop of Fig. 1 since $xex^{-1} = \phi(e)$ is a defining relation. Similarly, each edge of $\phi(e)$ can be slide along x, hence, for any $n > 0$, e can be slid along the edge path $\langle x, \ldots, x \rangle \equiv x^n$ at $*(0)$, to an edge path with edges in $\langle h_1^{\pm 1}, \ldots, h_n^{\pm 1} \rangle$. For each $k > 0$, let $M(k)$ be an integer such that each edge of $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ at $*(0)$ can be slid along x^k with image in $\mathrm{St}^{M(k)}(*(0))$. Let C be a compact subcomplex of $\tilde{X}$. C meets only finitely many levels of $\tilde{X}$. Choose integers A and B such that all levels above level B and all levels below level A (including levels A and B) miss C. Let $D = \mathrm{St}^{M(B-A)}(C)$.

Lemma 3.2. *If $e \in \{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ is an edge of $\tilde{X} - D$, and e lies in level Q, where $Q \geq A$, then e can be slid along x^t, for any $t > 0$ by a homotopy missing C.*

Proof. By the definition of $M(B-A)$, e can be slid along the first $B-A$ edges of x^t, missing C. The resulting edge path in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ is above level B, and hence can be slid along the remaining edges, missing C.

Next we prove: If H is 1-ended, then G is simply connected at ∞. Let D_1 be a compact set containing D such that if D_1 meets a copy of $\tilde{Y}$, the complement of D_1 in this copy of $\tilde{Y}$ has one unbounded path component. Let λ be an edge loop in $\tilde{X} - D_1$. If an edge of λ, corresponding to an element of $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ lies in a level below level A, slide it up to level A. Eliminating pairs of edges of the form

$\langle x, x^{-1}\rangle$ or $\langle x^{-1}, x\rangle$ we have λ is homotopic in $\tilde{X}-C$ to an edge loop λ_1, and each edge of λ_1 lies in level A or a level above level A. Furthermore since the homotopy of λ to λ_1 did not affect edges of λ above level A, if a is an edge of λ_1 corresponding to x, joining levels A and $A+1$, then the initial point of a misses D_1. Hence if B is a maximal subpath of λ_1 in the letters $\{h_1^{\pm 1}, \dots, h_n^{\pm 1}\}$, with image in level A of $\tilde{X}$, then the initial and end points of B are in $\tilde{X}-D_1$. B lies in a copy of $\tilde{Y}$, call it W, and by hypothesis $W-D_1$ is connected. Join the initial and end point of B by an edge path, $\bar{B}$, missing D_1, with edges in $\{h_1^{\pm 1}, \dots, h_n^{\pm 1}\}$. Since W is simply connected and misses C, B is homotopic rel$\{0,1\}$ to $\bar{B}$, by a homotopy missing C. Hence λ_1 is homotopic in $\tilde{X}-C$ to an edge loop, γ, such that each edge of γ from the set $\{h_1^{\pm 1}, \dots, h_n^{\pm 1}\}$ misses D_1 and lies in level A or a level above level A. By the definition of $D \subset D_1$ all edges of γ can be slid to a level above level B, by a homotopy missing C. The resulting edge loop lies in some copy of Y, missing C and hence is trivial there, by a homotopy missing C.

The next theorem can be done by standard techniques found in [6] or [10].

Theorem 3.2. *If H_1 and H_2 are 1-ended finitely presented subgroups of the finitely presented group G, such that $H_1 \cup H_2$ generates G, H_1 and H_2 are semi-stable at ∞ and $H_1 \cap H_2$ contains a finitely generated infinite group, then G is 1-ended and semi-stable at ∞.*

4. The main theorem

Our main theorem (see Section 1) is an easy corollary to Theorem 3.2 and the following:

Theorem 4.1. *Let H be a finitely presented subgroup of the finitely presented group G. If there is an element $x \in G$ such that, $H \cup \{x\}$ generates G, $xHx^{-1} \subset H$ and $G/N(H) \cong \mathbb{Z}$, then G is semi-stable at ∞ and 1 or 2-ended.*

Proof. If H is finite, then $xHx^{-1} = H$ and G is a group extension of a finite group by the integers. Theorem 3.3 of [10] shows G is 2-ended and semi-stable at ∞. Let $\langle h_1, \dots, h_n : r_1, \dots, r_a\rangle$ be a presentation for H. Let $P = \langle x, h_1, \dots, h_n : t_1, \dots, t_c\rangle$ be a presentation of G where $r_i \in \{t_1, \dots, t_c\}$ for $i \in \{1, \dots, a\}$, and $xh_ix^{-1}\alpha_i \in \{t_i, \dots, t_c\}$ where α_i is a word in the letters $\{h_1^{\pm 1}, \dots, h_n^{\pm 1}\}$. Let Y and X be the standard 2-complexes with $\pi_1(Y) = H$ and $\pi_1(X) = G$ obtained respectively from the above presentations. As in the proof of Theorem 3.1, let $*(0)$ be a vertex of $\tilde{X}$, and $*(n)$ the end point of the lift of the loop x^n, of X to $*(0) \in \tilde{X}$ for any integer n. Furthermore, define vertices of $\tilde{X}$ and copies of $\tilde{Y}$ which lie in $\tilde{X}$, to be in level n of $\tilde{X}$ as defined in Theorem 3.1. As base ray, we choose the proper edge path to ∞, $r : ([0, \infty), \{0\}) \to (\tilde{X}, *(0))$ such that r restricted to $[n, n+1]$, for $n \geq 0$, is the lift of

x to $*(n)$. For each $i>0$, let $M(i)$ be an integer such that each edge of $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ can be slid along x^i with image in $\mathrm{St}^{M(i)}(*(0))$.

Let C be a compact subcomplex of $\tilde{X}$. Choose integers A and B such that all levels above level B and all levels below level A (including levels A and B) miss C.

Lemma 4.2. *If $e \in \{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ is an edge of $\tilde{X} - \mathrm{St}^{M(B-A)}(C)$ and e lies in level Q, where $Q \geq A$, then e can be slid along x^t, for any $t \geq 0$, by a homotopy missing C.*

Proof. See Lemma 3.2.

If $h \in H$, then h has degree $i>0$ ($\deg(h)=i$) if i is the largest integer such that $h = x^i \bar{h} x^{-i}$ for some $\bar{h} \in H$. If no such largest integer exists, then $\deg(h)=\infty$.

If $\deg(h)=d<\infty$, then $x^d \bar{h} x^{-d} = h$ for some $\bar{h} \in H$. Hence for any $p \in \{0, 1, \ldots, d\}$, $x^p \hat{h} x^{-p} = h$ where $\hat{h} = x^{d-p} \bar{h} x^{(p-d)} \in H$, and we have:

Lemma 4.3. *If $\deg(h)=d \geq 0$, then for each $i \in \{0, 1, \ldots, d\}$ $h = x^i p_i x^{-i}$ for some $p_i \in H$.*

If (a, b) is a pair of vertices of $\tilde{X}$ and the covering transformation of $\tilde{X}$ that takes a to b corresponds to $h \in H$, then we say $\deg(a, b) = \deg(h)$. The following lemma is a key step in bridging the algebra and the desired geometry.

Lemma 4.4. *Assume E is a compact subcomplex of $\tilde{X}$, and Q is a copy of $\tilde{Y}$, in $\tilde{X}$. If W and L are distinct unbounded path components of $Q-E$, then one of the following two statements must hold:*

(i) *$W \times L$ contains a collection of pairs of vertices $(v_1, w_1), (v_2, w_2), \ldots$ such that all v_i are distinct and all w_i are distinct, and $\deg(v_i, w_i) \geq i$, or*

(ii) *there are finite sets of vertices $S \subset W$ and $T \subset L$ such that all pairs of vertices of $(W-S) \times (L-T)$ have degree less than some positive integer $N_1(W, L)$.*

Proof. Begin selecting pairs of vertices (v_i, w_i) satisfying the hypothesis of (i). If for some $i \geq 1$, (v_i, w_i) cannot be selected to satisfy this hypothesis, then either all pairs of vertices of $W \times L$ have degree less than or equal to $\max_{1 \leq j \leq i} \deg(v_j, w_j)$ or all pairs of vertices $(v, w) \in W \times L$ with degree $\geq i$ have $v \in \{v_1, \ldots, v_{i-1}\}$ or $w \in \{w_1, \ldots, w_{i-1}\}$. In the former situation $S=T=\emptyset$ and $N_1(W, L) = \max_{1 \leq j \leq i}\{\deg(v_j, w_j)\}$. In the latter situation $S = \{v_1, \ldots, v_{i-1}\}$, $T = \{w_1, \ldots, w_{i-1}\}$ and $N_1(W, L) = i$, completing the proof.

Since E is compact, it meets only a finite number of copies, Q, of $\tilde{Y}$. Also, for each such Q, $Q-E$ has only a finite number of unbounded path components. Thus, there are only a finite number of pairs (W, L) where W and L are distinct unbounded path components of $Q-E$.

We list these pairs as $(W_1, L_1), (W_2, L_2), \ldots, (W_p, L_p)$. If condition (i) of Lemma

4.4 holds for (W_j, L_j), let $S_j = T_j = \emptyset$ and $N_j = 0$. If condition (ii) of Lemma 4.4 holds for (W_j, L_j), let S_j and T_j be the finite sets of vertices defined in condition (ii) of Lemma 4.4 and let $N_j \equiv N_1(W_j, L_j)$, also defined in Lemma 4.4. Finally, let $S_1(E) = \bigcup_{j=1}^{p} (S_j \cup T_j)$ and let $N_1(E) = \max_{1 \le j \le p}\{N_j\}$. Note that $S_1(E)$ is a finite set.

The proof of the next lemma is analogous to that of Lemma 4.4 and left to the reader.

Lemma 4.5. *Assume E is a compact subcomplex of $\tilde{X}$ and Q is a copy of $\tilde{Y}$, in $\tilde{X}$. If W is an unbounded path component of $Q - E$ and v is vertex of $Q \cap E$ or a bounded path component of $Q - E$, then either:*

(i) *W contains a collection of distinct vertices $v_1, v_2, \dots$ such that* $\deg(v_i, v) \ge i$ *or,*

(ii) *there is a finite set of vertices $S \subset W$ such that for any vertex $w \in W - S$,* $\deg(w, v)$ *is less than or equal to some integer $N_2(W, v)$.*

A compact set E meets only a finite number of copies of $\tilde{Y}$, in $\tilde{X}$. Also if Q is a copy of $\tilde{Y}$ in $\tilde{X}$, then E union all bounded path components of $Q - E$ is compact. Hence there are only a finite number of pairs (W, v) where W is an unbounded path component of $Q - E$, and v is a vertex of either $Q \cap E$ or a bounded path component of $Q - E$. We list all such pairs as: $(W_1, v_1), (W_2, v_2), \dots, (W_p v_p)$. If condition (i) of Lemma 4.5 holds for (W_j, v_j), we let $S_j = \emptyset$, and $N_j = 0$. If condition (ii) of Lemma 4.5 holds for (W_j, v_j), we let S_j be the finite set of vertices defined in (ii), and $N_j = N_2(W_j, v)$ also defined in (ii) of Lemma 4.5. Finally, let $S_2(E) = \bigcup_{j=1}^{p} S_j$ and $N_2(E) = \mathrm{Max}_{1 \le j \le p}\{N_j\}$.

Lemma 4.6. *Assume E is a compact subset of $\tilde{X}$, and $Q \subset \tilde{X}$ is a copy of $\tilde{Y}$. If v and w are vertices of $(E \cap Q) \cup \{x \mid x$ is an element of a bounded path component of $Q - E)$, then either*

(i) $\deg(v, w)$ *is ∞, or*

(ii) *there is a positive integer $N_3(v, w)$ larger than* $\deg(v, w)$.

Proof. Trivial.

There are only a finite number of pairs (v, w) such that Q is a copy of $\tilde{Y}$ in $\tilde{X}$, and, v and w are vertices of $(Q \cap E) \cup \{x \mid x$ is an element of a bounded path component of $Q - E\}$. We list these pairs as $(v_1, w_1), (v_2, w_2), \dots, (v_p, w_p)$. If $\deg(v_i, w_i) = \infty$, let $N_i = 0$. If $\deg(v_i, w_i)$ is finite, let N_i be an integer larger than this number. Let $N_3(E) = \mathrm{Max}_{1 \le i \le p}\{N_i\}$.

Recall C is a compact subcomplex of $\tilde{X}$. A and B are integers such that C is below level B and above level A. Choose C_1 a compact subcomplex of $\tilde{X}$ containing $\mathrm{St}^{M(B-A)}(C)$ and such that if Q is a copy of $\tilde{Y}$, then the 1-skeleton of $Q - C_1$ is a union of unbounded path components. Let N_1 be the $N_1(C_1)$ defined following Lemma 4.4 let C_2 be compact and contain C_1 and $S_1(C_1)$. ($S_1(C_1)$ is defined following Lemma 4.4.) Furthermore choose C_2 such that the one skeleton of the compliment of C_2 in any copy of $\tilde{Y}$ is a union of unbounded path components.

Let N_2 be the $N_2(C_2)$ defined following Lemma 4.5. Let C_3 be a compact subcomplex of $\tilde{X}$ containing C_2 and $S_2(C_2)$. Furthermore, choose C_3 such that the 1-skeleton of the compliment of C_3 in any copy of $\tilde{Y}$ is a union of unbounded path components. Let N_3 be the $N_3(C_3)$, defined following Lemma 4.5. Let $N = \max\{N_1, N_2, N_3\}$. By the definitions of N_1, N_2, N_3 we have:

Lemma 4.7. *Let e be an edge of $\tilde{X}$ corresponding to an element of $\{h_1^{\pm 1}, \dots, h_n^{\pm 1}\}$ in level L of $\tilde{X}$ where $L \leq A - N$. If $k > 0$ is such that $\langle x^{-k}, e, x^k \rangle$ has end points in level A, then these end points lie in the same copy of $\tilde{Y}$, call it Q. (e can be slid along x^k to an edge path in the letters $\{h_1^{\pm 1}, \dots, h_n^{\pm 1}\}$ joining them.) Also exactly one of the following hold:*

(i) *Both end points lie in C_2, in which case the degree of this pair of vertices is infinite (by the definition of N_3).*

(iia) *The initial point of $\langle x^{-k}, e, x^k \rangle$ lies in C_2, and the end point of $\langle x^{-k}, e, x^k \rangle$ lies in W, an unbounded path component of $Q - C_2$.*

(iib) *The end point of $\langle x^{-k}, e, x^k \rangle$ lies in C_2, and the initial point of $\langle x^{-k}, e, x^k \rangle$ lies in W, an unbounded path component of $Q - C_2$.*

In both cases, (iia) *and* (iib), *W contains a collection $v_1, v_2, \dots$ of distinct vertices such that the* $\deg(v_i, v) \geq i$ *where v is the initial point of $\langle x^{-k}, e, x^k \rangle$ in case* (iia) *and v is the end point of $\langle x^{-k}, e, x^k \rangle$ in case* (iib). (See the definition of N_2.)

(iii) (i), (iia) *and* (iib) *do not hold and both end points of $\langle x^{-k}, e, x^k \rangle$ lie in the same unbounded path component, W, of $Q - C_1$.*

(iv) (i), (iia) *and* (iib) *do not hold and the end points of $\langle x^{-k}, e, x^k \rangle$ lie in different unbounded path components of $Q - C_1$. Call these path components W_1 and W_2. In this case $W_1 \times W_2$ contains pairs of vertices $(v_1, w_1)(v_2, w_2), \dots$ such that $v_i = v_j$ if and only if $i = j$ and $w_i = w_j$ if and only if $i = j$. Furthermore* $\deg(v_i, w_i) \geq i$ *by the definition of N_1.*

Proof. Clearly no two of (i), (iia),(iib), (iii) and (iv) can hold simultaneously. If (i), (iia), and (iib) fail, then the end points of $\langle x^{-k}, e, x^k \rangle$ must lie in $Q - C_1$ and either (iii) or (iv) must hold.

Let C_4 contain $\mathrm{St}^{M(B-A+N)}(C_3)$ and be a compact subcomplex of $\tilde{X}$. If l and s are the largest and smallest integers respectively such that $*(l)$ and $*(s)$ are in C_4 then assume, without loss, that $*(0) \in C_4$ and $*(i) \in C_4$ for $s \leq i \leq l$. By Theorem 2.1 of [10] it suffices to show: Let E be compact containing C. Then any edge loop, α, with image in $\tilde{X} - C_4$, and based on the proper ray to ∞, $r = (x, x, \dots)$ at $*(0)$, is homotopic rel. r to a loop in $\tilde{X} - E$, by a homotopy missing C.

Let $\alpha = \langle e_1, e_2, \dots, e_u \rangle$ be an edge loop based at $*(f)$, with image in $\tilde{X} - C_4$ (here we assume $f > 0$). Choose levels F and $\bar{F}$ of $\tilde{X}$ such that E and the image of α lie below level $\bar{F}$ and above level F.

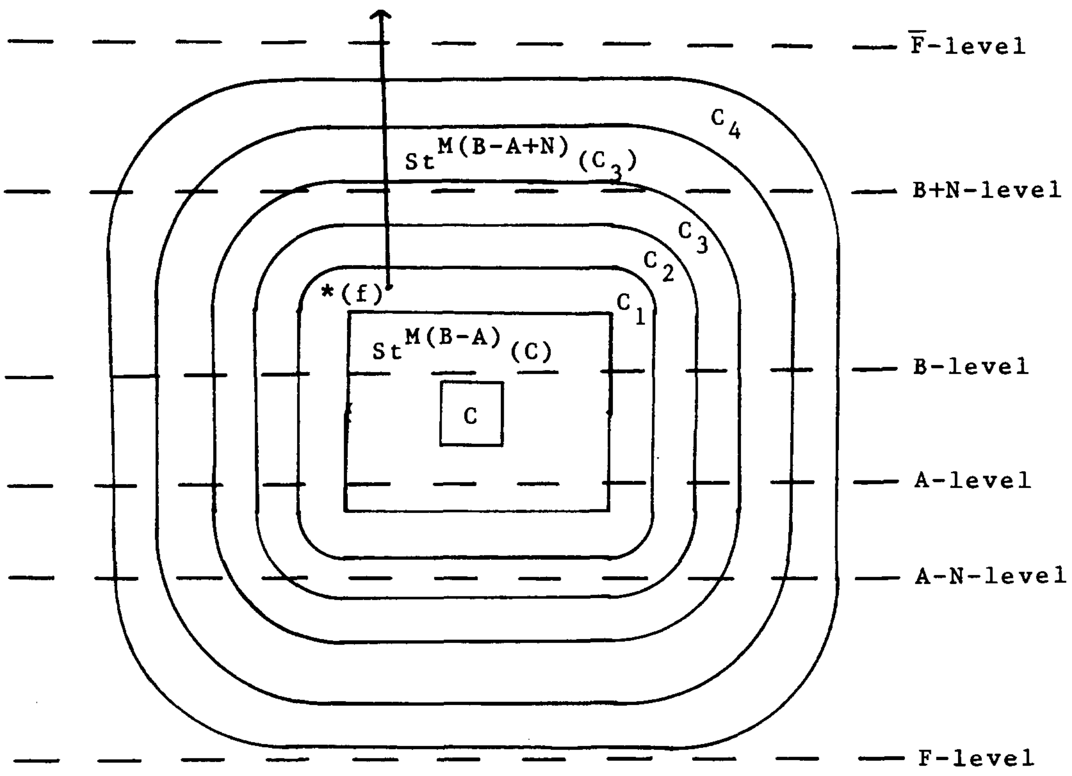

Fig. 2.

Lemma 4.8. *If the edge e_i of α corresponds to an element of $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ and e_i has image in level L where $L \geq A - N$, then e_i can be slid along x^t, for any $t > 0$, missing C. In particular e_i is homotopic* rel$\{0, 1\}$ *to $\langle x^{(\bar{F}-L)}, w_i, x^{-(\bar{F}-L)}\rangle$ where w_i is an edge path with letters in the set $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$.*

Proof. Since α misses C_4, e_i misses $\mathrm{St}^{M(B-A+N)}(C_3)$. Hence e_i misses $\mathrm{St}^{M(B-A+N)}(C_1)$. By the definition of M, e_i can be slid along x^{B-A+N} missing C_1. Since $L \geq A - N$, e_i can be slid to level B or above missing C_1. Since C is below level B, this sliding process can continue indefinitely, missing C.

Note. If the subpaths $x^{\bar{F}-L}$, and $x^{-(\bar{F}-L)}$ of $\langle x^{\bar{F}-L}, w_i, x^{-(\bar{F}-L)}\rangle$ pierce level A, they do so in vertices missing C_1.

If $e_i \in \{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ and e_i is in level L where $L < A - N$, then the end points of the edge path $\langle x^{-(A-L)}, e_i, x^{(A-L)}\rangle$ are in level A. Let the initial vertex of the edge path be v and the terminal vertex be w. Then by Lemma 4.7 exactly one of the five conclusions of Lemma 4.7 hold for e_i.

If (i), (iia), (iib), (iii) or (iv) of Lemma 4.7 holds we say e_i is respectively an edge of type (i), (iia), (iib), (iii) or (iv).

Recall e_i is an edge of $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ and e_i is in level $L \leq A - N$.

Lemma 4.9. *If e_i is to type* (i), *then e_i is homotopic* rel$\{0, 1\}$, *by a homotopy miss-*

ing C to the edge path $\langle x^{-(L-F)}, w_i, x^{l-F}\rangle$ where w_i is an edge path in the letters $\{h_1^{\pm1},\dots,h_n^{\pm1}\}$ and image in level F.

Proof. The initial and end points of $\langle x^{-(A-L)}, e_i, x^{(A-L)}\rangle$ are v and w respectively and lie in C_3 by the definition of type (i). By the definition of $N_3(\le N)$, (v, w) has infinite degree. By Lemma 4.3 there is an element w_i of H such that $x^{-(A-F)}w_i x^{A-F} = x^{-(A-L)}e_i x^{A-L}$ (here e_i takes on its representation in H). Hence as elements of $H, e_i = x^{L-F}w_i x^{-(L-F)}$. Geometrically this means the edge e_i and the edge path $\langle x^{-(L-F)}, w_i, x^{(L-F)}\rangle$ have the same initial and end points (see Fig. 3). w_i can be slid along x^{L-F} to y_i by a homotopy missing C (the homotopy lies below level L). y_i is an edge path in the letters $\langle h_1^{\pm1},\dots,h_n^{\pm1}\rangle$ and lies in level L. Furthermore $y_i e_i^{-1}$ is a loop in a copy of $\tilde{Y}$ that lies in level L. Hence y_i is homotopic rel$\{0,1\}$ to e_i by a homotopy in this copy of $\tilde{Y}$ which must miss C, completing the proof.

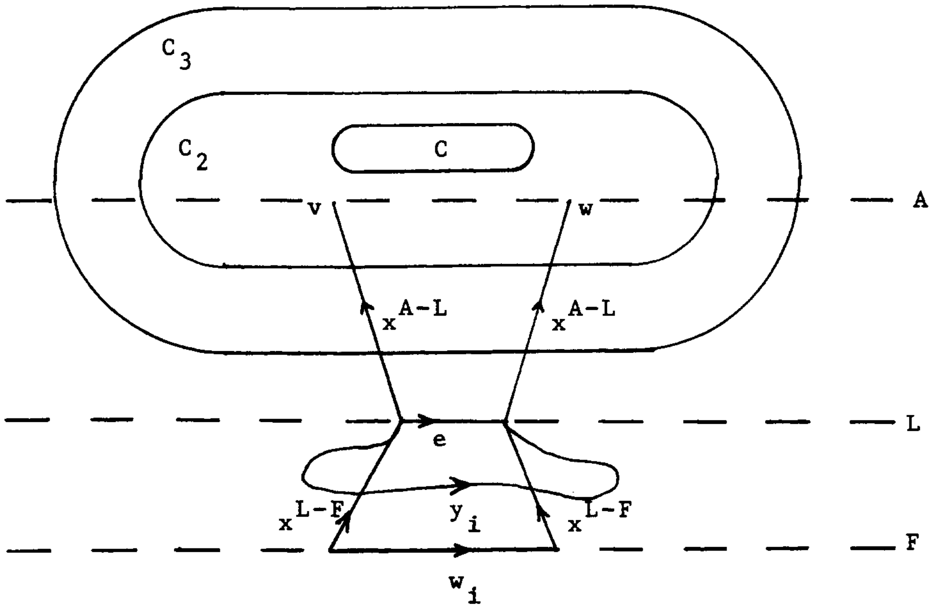

Fig. 3.

Lemma 4.10. *If e_i is an edge of type* (iia), *then e_i is homotopic* rel$\{0,1\}$, *by a homotopy missing C, to the edge path $\langle x^{-(L-F)}, a_i, x^{(\bar{F}-F)}, b_i, x^{-(\bar{F}-L)}\rangle$ where b_i and a_i are edges in the letters $\{h_1^{\pm1},\dots,h_n^{\pm1}\}$ in levels $\bar{F}$ and F respectively. Furthermore, the subpath $x^{-(\bar{F}-F)}$ misses E.*

Proof. We use Fig. 4 for guidance.

If e_i is slid along x^{A-L}, then the resulting edge path, g, lies in level A and is an edge path in the edges $\{h_1^{\pm1},\dots,h_n^{\pm1}\}$. The initial point, v, of g lies in C_2 and the end point, w, of g lies in W, an unbounded path component of $Q-C_2$, where Q is a copy of $\tilde{Y}$ in level A (by the definition of type (iia)). Furthermore (see 4.7(iia)) W contains a collection $v_1, v_2, \dots$ of distinct vertices of $\tilde{X}$, such that $\deg(v_i, v) \ge i$.

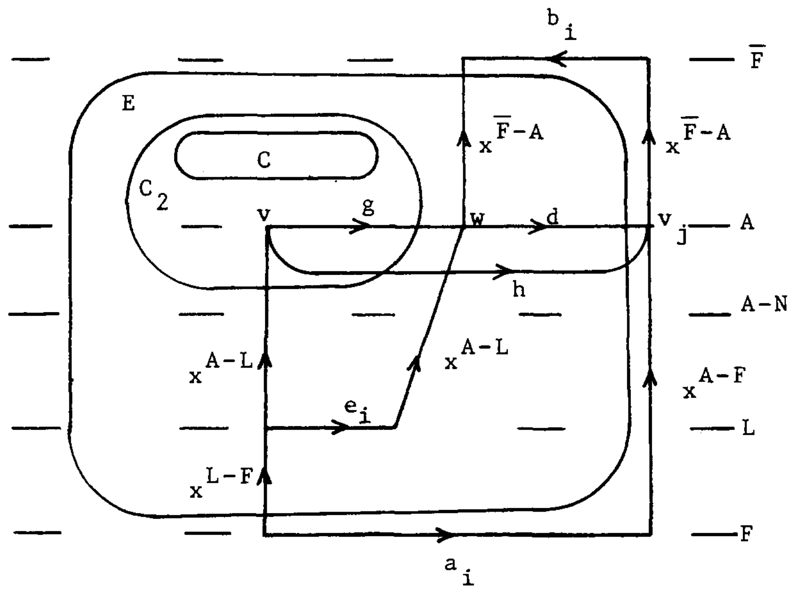

Fig. 4.

Since E is compact, there is only a finite collection of elements, t, in $\{v_1, v_2, \ldots\}$ such that the edge paths $x^{\bar{F}-A}$ and $x^{-(A-F)}$ at t meet E. Choose $j \geq A-F$ such that $x^{\bar{F}-A}$ at v_j and $x^{-(A-F)}$ at v_j miss E. Let d be an edge path from w to v_j in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$. Since w and v_j are in W, we assume d lies in W. Since $\deg(v_j, v) \geq A-F$ there is an edge path a_i in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ in the level F such that $\langle x^{-(A-F)}, a_i^{-1}, x^{A-F}\rangle$ has initial point v_j and end point v. Since C lies above level A, the slide of a_i along x^{A-F} to the edge path h, misses C. $h^{-1}gd$ forms a loop in Q. Since Q is a copy of $\tilde{Y}$ missing C, h is homotopic rel$\{0,1\}$ to gd, by a homotopy in Q and hence missing C. Thus $\langle x^{-(A-F)}, a_i^{-1}, x^{A-F}\rangle$ is homotopic rel$\{0,1\}$ to $\langle d^{-1}, g^{-1}\rangle$, by a homotopy missing C. Using the fact that C is above level A again, we have $\langle x^{-(A-L)}, e_i, x^{A-L}\rangle$ is homotopic to g by a homotopy missing C. Combining those two homotopies gives: $\langle x^{-(A-F)}, a_i^{-1}, x^{L-F}, e_i, x^{A-L}\rangle$ is homotopic rel$\{0,1\}$ to d^{-1}. Since d^{-1} has image in $W \subset \tilde{X} - C_2$, d^{-1} must miss C_1. By Lemma 4.2, d^{-1} can be slid along $x^{\bar{F}-A}$, missing C, to say b_i, an edge path in level $\bar{F}$ in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$. I.e. d^{-1} is homotopic rel$\{0,1\}$ to $\langle x^{\bar{F}-A}, b_i, x^{-(\bar{F}-A)}\rangle$ by a homotopy missing C. Combining this with the above homotopy gives $\langle x^{-(A-F)}, a_i^{-1}, x^{L-F}, e_i, x^{A-L}\rangle$ is homotopic rel$\{0,1\}$ to $\langle x^{\bar{F}-A}, b_i, x^{-(\bar{F}-A)}\rangle$ by a homotopy missing C. Hence e_i is homotopic rel$\{0,1\}$ to $\langle x^{-(L-F)}, a_i, x^{\bar{F}-F}, b_i, x^{-(\bar{F}-L)}\rangle$ by a homotopy missing C.

In direct analogy to Lemma 4.10 we have Lemma 4.11.

Lemma 4.11. *If e_i is an edge of type* (iib), *then e_i is homotopic* rel$\{0,1\}$, *by a homotopy missing C, to the edge path $\langle x^{\bar{F}-L}, a_i, x^{-(\bar{F}-F)}, b_i, x^{L-F}\rangle$ where a_i and b_i are edge paths in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ in levels $\bar{F}$ and F respectively. Furthermore the subpath $x^{-(\bar{F}-F)}$ misses E.*

Lemma 4.12. *If e_i is an edge of type* (iii), *then e_i is homotopic* rel$\{0,1\}$, *by a homotopy missing C, to the edge path $\langle x^{\bar{F}-L}, w_i, x^{-(\bar{F}-L)}\rangle$ where w_i is an edge path in the edges $\langle h_1^{\pm 1},\ldots,h_n^{\pm 1}\rangle$ and image in level $\bar{F}$.*

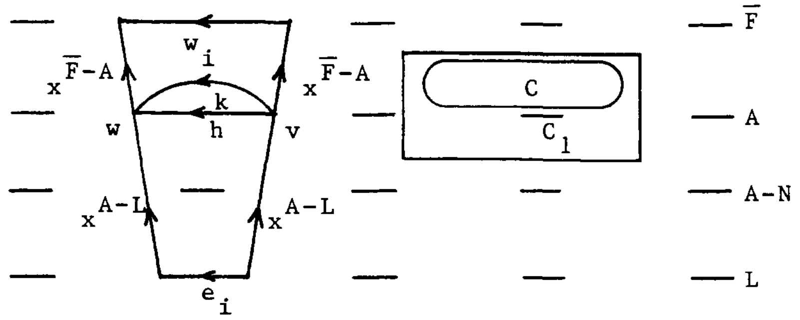

Fig. 5.

Proof. By the definition of type (iii), v and w lie in the same unbounded path component, W, of $Q-C_1$ where Q is a copy of $\tilde{Y}$ in level A. Let h be an edge path in the letters $\{h_1^{\pm 1},\ldots,h_n^{\pm 1}\}$ in W from v to w. Let k be the slide of e_i along x^{A-L}. Then $\langle h,k^{-1}\rangle$ is a loop in Q and is hence homotopically trivial in Q. Since C is above level A, Q misses C and this homotopy misses C. Furthermore, the slide of e_i along x^{A-L} misses C. Hence e_i is homotopic rel$\{0,1\}$ to $\langle x^{A-L},k,x^{-(A-L)}\rangle$ by a homotopy missing C. This implies e_i is homotopic rel$\{0,1\}$ to $\langle x^{A-L},h,x^{-(A-L)}\rangle$ by a homotopy missing C. But h misses C_1 and lies in level A, so by Lemma 4.2 h can be slid along $x^{\bar{F}-A}$, missing C, to say w_i. Replace h by $\langle x^{\bar{F}-A},w_i,x^{-(\bar{F}-A)}\rangle$ in $\langle x^{A-L},h,x^{-(A-L)}\rangle$.

Lemma 4.13. *If e_i is an edge of type* (iv), *then e_i is homotopic* rel$\{0,1\}$, *by a homotopy missing C, to an edge path, $\langle x^{\bar{F}-L},a_i,x^{-(\bar{F}-F)},b_i,x^{\bar{F}-F},d_i,x^{-(\bar{F}-L)}\rangle$, where a_i, b_i and d_i are edge paths in the letters $\langle h_1^{\pm 1},\ldots,h_n^{\pm 1}\rangle$, a_i and d_i are in level $\bar{F}$, b_i is in level F and the subpaths $x^{-(\bar{F}-F)}$ and $x^{\bar{F}-F}$ miss E.*

The proof of this Lemma is analogous to that of Lemma 4.10 and we outline it, using Fig. 6 for a guide.

By the definition of (iv), v and w lie in different unbounded path components of $Q-C_1$, where Q is a copy of $\tilde{Y}$. Call these path components R and S respectively. $R\times S$ contains collection of pairs of vertices $(v_1,w_1,v_2,w_2),\ldots$ such that $\deg(v_i,w_i)$ is $\geq i$. Select (v_j,w_j) such that $x^{\bar{F}-A}$ at v_j and at w_j miss E, $x^{-(A-F)}$ at v_j and at w_j miss E and such that $j\geq A-F$. Since $j\geq A-F$, there is an edge path b_i in the letters $\{h_1^{\pm 1},\ldots,h_n^{\pm 1}\}$ in level F such that $\langle x^{-(A-F)},b_i,x^{A-F}\rangle$ has initial point v_j and end point w_j. Let y and z be edge paths in the edges $\{h_1^{\pm 1},\ldots,h_n^{\pm 1}\}$ from w_j to w, and from v to v_j respectively. Furthermore, select y to have image in the path component S and z to have image in R. Let g be the slide of e_i along z^{A-F}, d_i the slide of y along $x^{\bar{F}-A}$, and a_i the slide of z along $x^{\bar{F}-A}$. Each of these homotopies miss

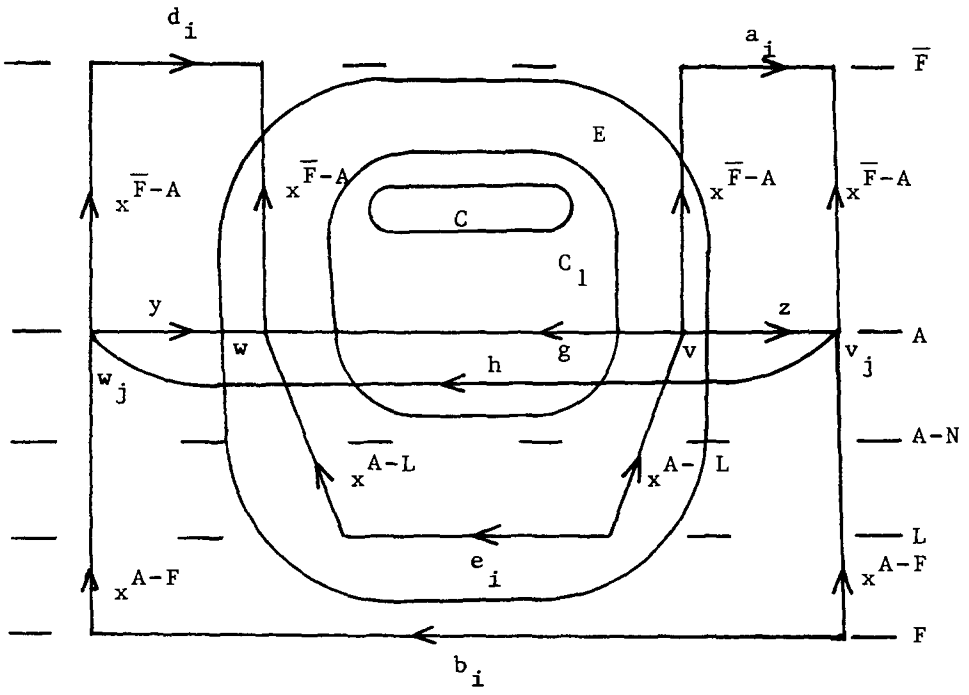

Fig. 6.

C, since the first homotopy occurs below level A, and since y and z miss C_1 and lie in level A. Let h be the slide of b_i along x^{A-F}. This homotopy has image below level A. $\langle h, y, g^{-1}, z\rangle$ is a loop in Q and hence is homotopically trivial there. Combining the homotopies of $\langle x^{-(A-F)}, b_i, x^{A-F}\rangle$ to h, h to $\langle z^{-1}, g, y^{-1}\rangle$, y to $\langle x^{\bar{F}-A}, d_i, x^{-(\bar{F}-A)}\rangle$, z to $\langle x^{\bar{F}-A}, a_i, x^{-(\bar{F}-A)}\rangle$ and e_i to $\langle x^{A-F}, g, x^{-(A-F)}\rangle$ as in Fig. 3 gives the desired homotopy.

If e_i is not an edge of type (i), (iia), (iib), (iii), or (iv), then it is either x, x^{-1}, or an element $\{h_1^{\pm1}, \dots, h_n^{\pm1}\}$ that lies in level $Q-N$ or above, in which case we say e_i is of type (0).

By Lemma 4.8, if e_i is of type (0), e_i is homotopic rel$\{0,1\}$, by a homotopy missing C, to the edge path $\langle x^{\bar{F}-L}, w_i, x^{-(\bar{F}-L)}\rangle$ where w_i is a word in the letters $\langle h_1^{\pm1}, \dots, h_n^{\pm1}\rangle$ and image in level F.

We have shown that each edge e_i of $\alpha = \langle e_1, \dots, e_u\rangle$ is homotopic rel$\{0,1\}$ to an edge path $\langle x^k, \beta, x^l\rangle$ where β misses E (see Figs. 3-6). Furthermore if e_i is of type (0), (iia), (iib), (iii) or (iv), then x^k and x^l can only pierce level A in a point missing C_1. If e_i is of type (i), x^k and x^l lie below level A.

Replace each e_i of α of the form $\{h_1^{\pm1}, \dots, h_n^{\pm1}\}$ by the appropriate edge path of the form $\langle x^k, \beta, x^l\rangle$ where β misses E. The resulting edge path, call it δ is homotopic rel$\{0,1\}$ to α by a homotopy missing C. After eliminating edges of the form $\langle x, x^{-1}\rangle$ and $\langle x^{-1}, x\rangle$, we have an edge loop that can be represented as $\langle w_1, w_2, \dots, w_p\rangle$ where either

(1) $w_i \in \{h_1^{\pm1}, \dots, h_n^{\pm1}\}$ and has image in level F or $\bar{F}$ (which misses E), or

(2) $w_i = x^{\pm(\bar{F}-F)}$.

Since no edge of α of the form $x^{\pm1}$ meets C_1, our above analysis of the replacement edge paths for edges of α implies: If $w_i = x^{\pm(\bar{F}-F)}$ and w_i meets E, then w_i pierces level A in a point missing C_1.

It suffices to show:

Lemma 4.14. *If $w = x^{\bar{F}-F}$ has initial point in level F and end point in level $\bar{F}$, and w pierces level A in a point missing C_1, then w is homotopic* rel$\{0,1\}$ *by a homotopy missing C to an edge path missing E.*

Proof. Let a be the initial point of w and b the end point of w. Let v be the point at which w pierces level A.

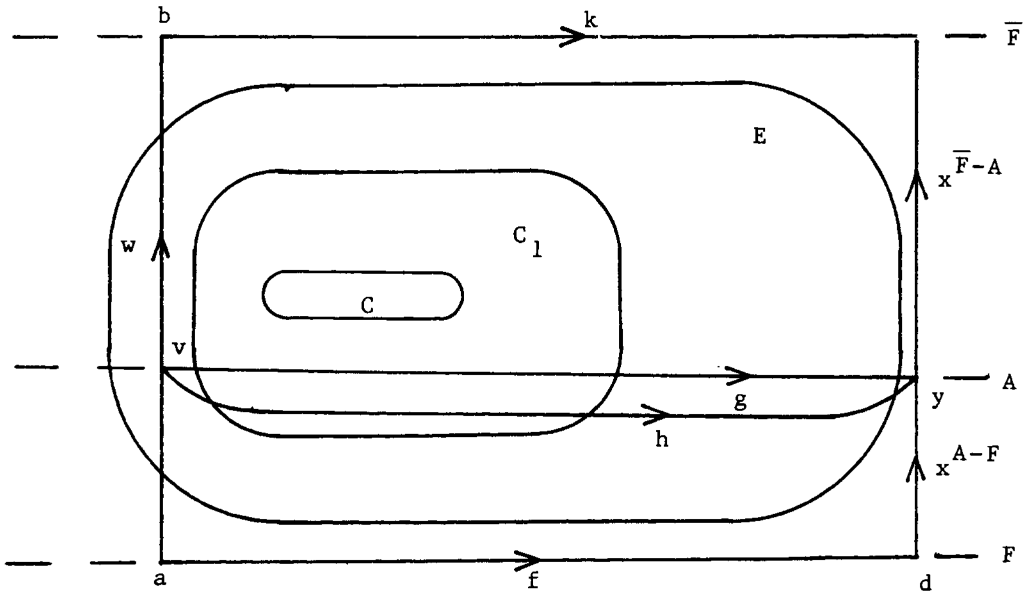

Fig. 7.

Choose an edge path f in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ from a to a point d such that $x^{\bar{F}-F}$ at d misses E. Let y be the end point of x^{A-F} at d. By sliding f along x^{A-F} we obtain an edge path, g, in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ from v to y. Hence v and y are in the same copy Q of $\tilde{Y}$. Also v and y miss C_1. If v and y are in the same unbounded path component, W, of $Q - C_1$, then select an edge path, h, from v to y with image in W in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$. h is homotopic to g by a homotopy missing C (Q lies in level A and C is above level A). Since h misses C_1, it can be slid along $x^{\bar{F}-A}$ to k by a homotopy missing C. Combining the homotopies of $\langle x^{-(A-F)}, f, x^{A-F}\rangle$ to g, g to h, and h to $\langle x^{\bar{F}-A}, k, x^{-(\bar{F}-A)}\rangle$ defines a homotopy rel$\{0,1\}$ of w to $\langle f, x^{\bar{F}-F}, k^{-1}\rangle$. The image of this homotopy misses C, and $\langle f, x^{\bar{F}-\bar{F}}, k^{-1}\rangle$ misses E as desired. If v and y are in different unbounded path components W_1 and W_2 of $Q - C$, then by the definition of $N_1(<N)$, $W_1 \times W_2$ contains pairs of vertices $(v_1, w_1), (v_2, w_2), \ldots$ such that $\deg(v_i, w_i) \geq i$. Choose $j \geq A - F$ and such that $x^{\bar{F}-A}$ and $x^{-(A-F)}$ at v_j and w_j miss E. Choose edge paths u_1 and u_2 in the letters $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ in W_2 and W_1, respectively, from y to w_j and v to v_j, respec-

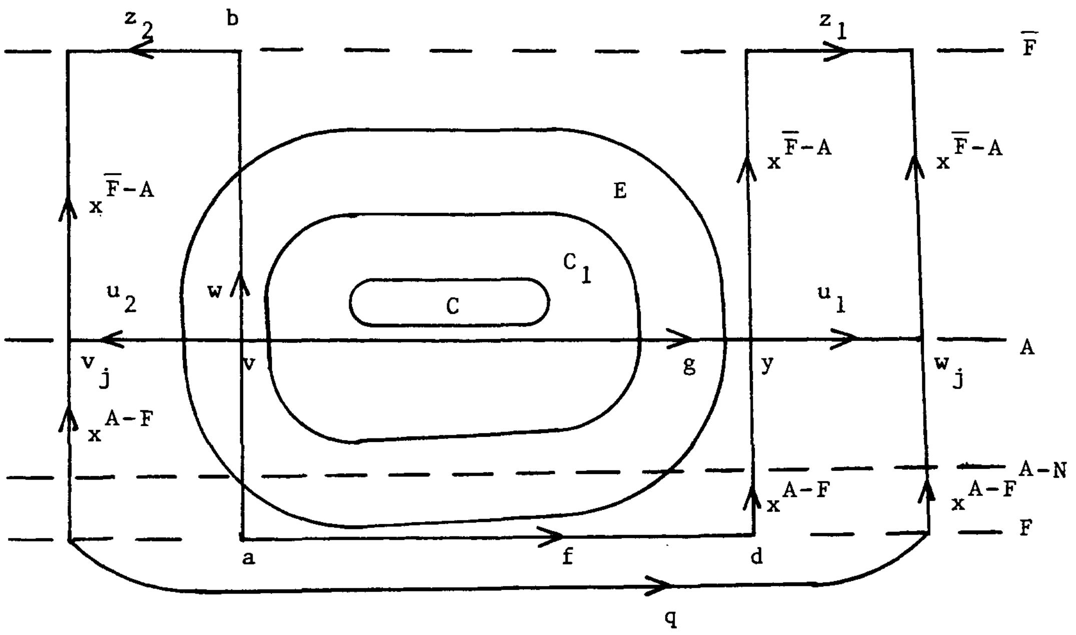

Fig. 8.

tively. Let z_1 and z_2 be the slides of u_1 and u_2, respectively, along $x^{\bar{F}-A}$. In an argument completely analogous to that used in Lemma 4.13, we see $x^{\bar{F}-F}$ at a is homotopic rel$\{0,1\}$, by a homotopy missing C to $\langle f, x^{\bar{F}-F}, z_1, x^{-(\bar{F}-F)}, q^{-1}, x^{\bar{F}-F}, z_2^{-1}\rangle$, which misses E.

Finally we note that the homotopy of α to δ was rel$\{0,1\}$ and δ is changed by homotopies rel$\{0,1\}$ or by cancelling edges of the form $\langle x, x^{-1}\rangle$, or $\langle x^{-1}, x\rangle$. Since our base ray r is $\langle x, x, \ldots\rangle$ at $*(0)$, these homotopies are all rel. r. At this point we see $H^2(G:\mathbb{Z}G)$ is free abelian (see Section 1). We omit the proof that G is 1-ended and comment that the 1-endedness of G can be shown by techniques similar to those already exhibited.

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